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Fracture analysis of circular-arc interface cracks in piezoelectric materials

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Abstract

The anti-plane problem of N arc-shaped interfacial cracks between a circular piezoelectric inhomogeneity and an infinite piezoelectric matrix is investigated by means of the complex variable method. Cracks are assumed to be permeable and then explicit expressions are presented, respectively, for the electric field on the crack faces, the complex potentials in media and the intensity factors near the crack-tips. As examples, the corresponding solutions are obtained for a piezoelectric bimaterial system with one or two permeable arc-shaped interfacial cracks, respectively. Additionally, the solutions for the cases of impermeable cracks also are given by treating an impermeable crack as a particular case of a permeable crack. It is shown that for the case of permeable interfacial cracks, the electric field is jumpy ahead of the crack tips, and its intensity factor is always dependent on that of stress. Moreover all the field singularities are dependent not only on the applied mechanical load, but also on the applied electric load. However, for the case of a homogeneous material with permeable cracks, all the singular factors are related only to the applied stresses and material constants.

Keywords: Piezoelectric media; Anti-plane strain; Inhomogeneity; Permeable interface crack

1. Introduction

With increasingly wide application of piezoelectric-fiber composites in engineering, it is of both theoretical and practical importance to study the arc-shaped crack problems in piezoelectric materials. Indeed, the problem has been received considerable interest in recent years, and several important solutions have been presented by Zhong and Meguid (1997a,b) for the case of a circular arc-crack in a homogeneous piezoelectric material, and by Zhong and Meguid (1997a,b), and also Deng and Meguid (1999) for the case of a circular-arc interface crack in piezoelectric bimaterials. In their works the cracks are assumed to be impermeable. Recently, Gao and Fan (1999) investigated a similar problem to the above works, but the crack is assumed to be permeable. However, it should be noted that in Gao and Fan's work (1999), no

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solutions were presented for the electric field on the crack faces. In fact, the solutions are very important to the study of electric-boundary-condition evolution on crack faces. More and more findings show that a mathematical crack in piezoelectric media behaves more like a permeable slit in a low-level loading. With the increasing of the applied loading, the intensity of electric field on crack faces varies, and as a result the crack faces may evolve to be impermeable, semi-impermeable, conductive or more complicated cases. Although, up to now, it seems that one has yet not known much about the critical conditions of the above evolution, the study for the case of permeable cracks is the first step to the other complicated cases.

In this paper we will conduct an explicit and systemic analysis for N circular-arc interface cracks between a circular inhomogeneity and an infinite matrix. Considering that the cracks may emerge between piezoelectric fibers and the matrix, on the surface of an elastic inclusion in piezoelectric materials, or within a homogeneous piezoelectric medium, etc., both the inhomogeneity and matrix are assumed to be piezoelectric in the present analysis, in order to obtain a unity solution. Compared with the established literature on the topic, the development of this work includes: the present analysis is valid not only to permeable crack model, also to impermeable crack model; explicit results are presented for the electric field on crack faces, and closed-form solutions for the case of two arc-shaped interface cracks are at the first time obtained as one of examples. Moreover, the final solutions are expressed only by the applied loads and a bimaterial matrix **M**, and thus it becomes easy to observe the coupling effects taking place between mechanical and electric fields.

Below is the plan of this work. Following the brief introduction, Section 2 outlines the basic equations to be need in the later sections, and then the electric field on crack faces and the complex potentials in media are derived in Sections 3 and 4, respectively. Given in Section 5 are the expressions of field intensity factors. As examples, explicit solutions for the case of one crack, two cracks and without crack are obtained in Section 6, respectively. Finally, Section 7 concludes the work.

2. Basic equations

In a fixed Cartesian coordinate system x_j ($j = 1$–3), the general equations governing the three-dimensional theory of piezoelectricity can be written as (Pak, 1990)

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k, \tag{1}$$

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \tag{2}$$

$$2\gamma_{ij} = u_{j,i} + u_{i,j}, \quad E_i = -\varphi_{,i} \quad (i, j, k, l = 1\text{–}3), \tag{3}$$

where σ_{ij}, D_i, γ_{ij}, u_i, E_i and φ are the components of stress, electric displacement, strain, displacement, electric field and electric potential, respectively; c_{ijkl}, e_{kij} and ε_{ik} stand for elastic constants, piezoelectric constants and dielectric constants, respectively; a comma indicates partial derivative.

Consider a circular piezoelectric inhomogeneity partially unbonded in an infinite piezoelectric matrix, which is subjected to anti-plane mechanical loads as well as inplane electric loads at infinity, as shown in Fig. 1. Assume that both the inhomogeneity and matrix are transversely isotropic with respect to the x_3-axis, and $x_1 - x_2$ is the isotropic plane, in which the regions occupied by the inhomogeneity and matrix are denoted by s_1 and s_2, respectively. The interface between s_1 and s_2 is denoted by L ($L = L_c + L_b$), where L_c represents the interface cracks; and L_b, the bonded part. In addition, the cracks are assumed to be free of force and external charges, but filled with air or vacuum.

In this case, the displacement u_i and the electric potential φ have the form

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2), \quad \varphi = \varphi(x_1, x_2). \tag{4}$$

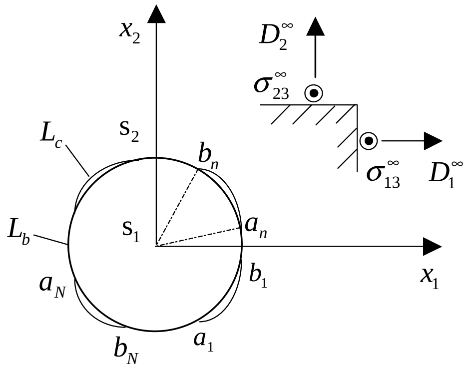

Fig. 1. N interfacial cracks between two piezoelectric materials.

Thus, (2) can be simplified, by substituting (3) with (4) into (1), as

$$\sigma_{3k} = c_{44}\frac{\partial u_3}{\partial x_k} + e_{15}\frac{\partial \varphi}{\partial x_k},$$

$$D_k = e_{15}\frac{\partial u_3}{\partial x_k} - \varepsilon_{11}\frac{\partial \varphi}{\partial x_k} \quad (k = 1, 2). \tag{5}$$

Inserting (5) into (2) results in

$$\begin{aligned} &c_{44}\nabla^2 u_3 + e_{15}\nabla^2\varphi = 0, \\ &e_{15}\nabla^2 u_3 - \varepsilon_{11}\nabla^2\varphi = 0, \end{aligned} \tag{6}$$

where $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the two-dimensional Laplacian operator.

Due to $e_{15}^2 + c_{44}\varepsilon_{11} \neq 0$, one has from (6) that

$$\nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0. \tag{7}$$

The general solution of (7) can be expressed as

$$u_3(x_1, x_2) = 2\,\mathrm{Im}\,\omega_1(z), \quad \varphi(x_1, x_2) = 2\,\mathrm{Im}\,\omega_2(z), \quad z = x_1 + \mathrm{i}x_2, \tag{8}$$

where Im stands for the imaginary part, and $\omega_k(z)$ are two complex functions.

Using (5), (3) and (8), the stress σ_{3j}, the electric displacement D_j and the electric field E_j can be determined by the following equations

$$\begin{aligned} &\sigma_{32} + \mathrm{i}\sigma_{31} = 2\sum_{k=1}^{2} b_{1k}\Omega_k(z), \\ &D_2 + \mathrm{i}D_1 = 2\sum_{k=1}^{2} b_{2k}\Omega_k(z), \quad E_2 + \mathrm{i}E_1 = -2\Omega_2(z), \end{aligned} \tag{9}$$

where $\Omega_k(z) = \mathrm{d}\omega_k(z)/\mathrm{d}z$, and b_{jk} are the elements of an elastic matrix:

$$[b_{jk}] = \mathbf{B} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}, \tag{10}$$

whose inverse matrix is

$$\mathbf{\Lambda} = \mathbf{B}^{-1} = \frac{1}{e_{15}^2 + c_{44}\varepsilon_{11}} \begin{bmatrix} \varepsilon_{11} & e_{15} \\ e_{15} & -c_{44} \end{bmatrix}. \tag{11}$$

Introducing a polar coordinate system (r, θ), the field variables can be rewritten as

$$\begin{aligned} &\sigma_{3\theta} + \mathrm{i}\sigma_{3r} = 2\mathrm{e}^{\mathrm{i}\theta} \sum_{k=1}^{2} b_{1k}\Omega_k(z), \quad z = r\mathrm{e}^{\mathrm{i}\theta}, \\ &D_\theta + \mathrm{i}D_r = 2\mathrm{e}^{\mathrm{i}\theta} \sum_{k=1}^{2} b_{2k}\Omega_k(z), \\ &E_\theta + \mathrm{i}E_r = -2\mathrm{e}^{\mathrm{i}\theta}\Omega_2(z), \quad u_3(r,\theta) = 2\,\mathrm{Im}\,\omega_1(z). \end{aligned} \tag{12}$$

From (12), one has

$$\begin{aligned} &\sigma_{3r} = 2\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\theta} \sum_{k=1}^{2} b_{1k}\Omega_k(z)\right], \\ &D_r = 2\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\theta} \sum_{k=1}^{2} b_{2k}\Omega_k(z)\right], \\ &E_r = -2\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\theta}\Omega_2(z)\right], \\ &E_\theta = -2\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\theta}\Omega_2(z)\right], \\ &u_3' = 2\,\mathrm{Re}[z\Omega_1(z)], \end{aligned} \tag{13}$$

where Re means taking the real part, and $u_3' = \partial u_3(r,\theta)/\partial\theta$.

From (13) the field variables can be expressed, in the form of vectors, as

$$\begin{Bmatrix} \sigma_{3r} \\ D_r \end{Bmatrix} = \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{i}}\left[B\mathbf{F}(z) - \frac{1}{\mathrm{e}^{2\mathrm{i}\theta}}\overline{B}\overline{\mathbf{F}(z)}\right], \quad \begin{Bmatrix} u'/r \\ -E_\theta \end{Bmatrix} = \mathrm{e}^{\mathrm{i}\theta}\left[\mathbf{F}(z) + \frac{1}{\mathrm{e}^{2\mathrm{i}\theta}}\overline{\mathbf{F}(z)}\right], \tag{14}$$

where

$$\mathbf{F}(z) = [\Omega_1(z), \Omega_2(z)]^{\mathrm{T}}.$$

On L, noting that $z = R\mathrm{e}^{\mathrm{i}\theta}$ and $\bar{z} = R^2/z$, (14) becomes

$$\begin{Bmatrix} \sigma_{3r} \\ D_r \end{Bmatrix}_L = \frac{z}{\mathrm{i}R}\left[B\mathbf{F}(z) - \frac{R^2}{z^2}\overline{B}\overline{\mathbf{F}(z)}\right], \quad \begin{Bmatrix} u_3'/R \\ -E_\theta \end{Bmatrix}_L = \frac{z}{R}\left[\mathbf{F}(z) + \frac{R^2}{z^2}\overline{\mathbf{F}(z)}\right]. \tag{15}$$

Define

$$\mathbf{T} = \frac{\mathrm{i}R}{z}\begin{Bmatrix} \sigma_{3r} \\ D_r \end{Bmatrix}, \quad \mathbf{U} = \frac{1}{z}\begin{Bmatrix} u_3' \\ -RE_\theta \end{Bmatrix}, \quad \mathbf{G}(z) = \frac{R^2}{z^2}\overline{\mathbf{F}}\left(\frac{R^2}{z}\right). \tag{16}$$

Then, we have from (15) and (16) that

$$\mathbf{T} = \mathbf{B}\mathbf{F}(z) - \mathbf{B}\mathbf{G}(\bar{z}) \quad \text{on } L, \tag{17}$$

$$\mathbf{U} = \mathbf{F}(z) + \mathbf{G}(\bar{z}) \quad \text{on } L, \tag{18}$$

where two complex functions $\mathbf{F}(z)$ and $\mathbf{G}(z)$ will be determined by the following boundary conditions (Parton, 1976):

$$\sigma_{3r}^{(1)} = 0 \quad \text{on } L_c, \tag{19}$$

$$u_3'^{(1)} = u_3'^{(2)} \quad \text{on } L_b, \tag{20}$$

$$\sigma_{3r}^{(1)} = \sigma_{3r}^{(2)}, \quad D_r^{(1)} = D_r^{(2)}, \quad E_\theta^{(1)} = E_\theta^{(2)} \quad \text{on } L = L_c + L_b. \tag{21}$$

3. Electric boundary value on the crack faces

In this section we study the electric field on the crack faces. At first, let us examine the nature of $\mathbf{F}(z)$ and $\mathbf{G}(z)$. For the region s_1, $\mathbf{F}_1(z)$ and $\mathbf{G}_1(z)$ have no singular point, and thus both have the form

$$\mathbf{F}_1(z) = \mathbf{F}_{10}(z), \quad \mathbf{G}_1(z) = \mathbf{G}_{10}(z), \tag{22}$$

where $\mathbf{F}_{10}(z)$ and $\mathbf{G}_{10}(z)$ are two analytical functions, respectively, in s_1 and s_2.

For the region s_2, $\mathbf{F}_2(z)$ has the form of

$$\mathbf{F}_2(z) = \mathbf{A}_0 + \frac{\mathbf{A}_{-1}}{z} + \mathbf{0}\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty, \tag{23}$$

where $\mathbf{A}_{-1}$ is a constant vector related to the resultant force P'_{30} and the sum of net charge Q'_0 on the whole circular-hole rim. According to the equilibrium condition of the inhomogeneity, obviously $P'_{30} = Q'_0 = 0$. This indicates $\mathbf{A}_{-1} = \mathbf{0}$; $\mathbf{A}_0$ is a known constant which is related to the applied loads by

$$\mathbf{B}_2\mathbf{A}_0 = \mathbf{p}^\infty, \quad 2\mathbf{p}^\infty = [\sigma_{32}^\infty + \mathrm{i}\sigma_{31}^\infty, D_2^\infty + \mathrm{i}D_1^\infty]^{\mathrm{T}}. \tag{24}$$

Thus, (23) can be rewritten as

$$\mathbf{F}_2(z) = \mathbf{A}_0 + \mathbf{F}_{20}(z), \tag{25}$$

where $\mathbf{F}_{20}(z) = \mathrm{O}(z^{-2})$, which is analytical in s_2.

Noting (16.3) and (23), $\mathbf{G}_2(z)$ has the from of

$$\mathbf{G}_2(z) = \frac{R^2}{z^2}\overline{\mathbf{A}_0} + \mathbf{G}_{20}(z), \tag{26}$$

where $\mathbf{G}_{20}(z)$ is a homolorphic function in s_1.

On the crack faces, the stress-free boundary condition requires

$$\mathbf{T} = \mathbf{i}_2\mathrm{i}e^{-\mathrm{i}\theta}D_r^0 \quad \text{on } L_c, \tag{27}$$

where $\mathbf{i}_2 = [0, 1]^{\mathrm{T}}$; D_r^0 is a unknown function. Obviously, the key task is to determine D_r^0. This will be done through the following procedure:

Firstly, one has from (21) that $\mathbf{T}_1 = \mathbf{T}_2$ on L, which leads to, from (17), that

$$\mathbf{B}_1\mathbf{F}_1^+(t) - \mathbf{B}_1\mathbf{G}_1^-(t) = \mathbf{B}_2\mathbf{F}_2^-(t) - \mathbf{B}_2\mathbf{G}_2^+(t), \quad t \in L. \tag{28}$$

Substituting (22)–(26) into (28) yields

$$[\mathbf{B}_1\mathbf{F}_{10}(t) + \mathbf{B}_2\mathbf{G}_{20}(t) - \mathbf{p}^\infty]^+ - \left[\mathbf{B}_2\mathbf{F}_{20}(t) + \mathbf{B}_1\mathbf{G}_{10}(t) - \overline{\mathbf{p}^\infty}\frac{R^2}{t^2}\right]^- = \mathbf{0}, \quad t \in L. \tag{29}$$

From (29) we have (Muskhelishvili, 1975)

$$\begin{aligned} &\mathbf{B}_1\mathbf{F}_{10}(z) + \mathbf{B}_2\mathbf{G}_{20}(z) - \mathbf{p}^{\infty} = \mathbf{0}, \quad z \in s_1, \\ &\mathbf{B}_2\mathbf{F}_{20}(z) + \mathbf{B}_1\mathbf{G}_{10}(z) - \overline{\mathbf{p}^{\infty}}\frac{R^2}{t^2} = \mathbf{0}, \quad z \in s_2. \end{aligned} \tag{30}$$

Secondly, introduce two vectors $\Delta\mathbf{U}(t)$ and $\mathbf{T}(t)$ as

$$\Delta\mathbf{U}(t) = \mathbf{U}_1(t) - \mathbf{U}_2(t), \tag{31}$$

$$\mathbf{T}(t) = \mathbf{B}_1\mathbf{F}_1^+(t) - \mathbf{B}_1\mathbf{G}_1^-(t), \quad t \in L. \tag{32}$$

Substituting (18) into (31), and then using (30) leads to

$$\Delta\mathbf{U}(t) = \mathbf{K}^+(t) - \mathbf{K}^-(t), \tag{33}$$

where

$$\mathbf{K}(z) = \begin{cases} \mathbf{H}\mathbf{B}_1\mathbf{F}_{10}(z) - 2\mathbf{B}_2^{-1}\mathbf{p}^{\infty}, & z \in s_1, \\ \mathbf{H}\mathbf{B}_2\mathbf{F}_{20}(z) - (\mathbf{B}_1^{-1} - \mathbf{B}_2^{-1})\overline{\mathbf{p}^{\infty}}\frac{R^2}{z^2}, & z \in s_2, \end{cases} \tag{34}$$

$$\mathbf{H} = \mathbf{B}_1^{-1} + \mathbf{B}_2^{-1}. \tag{35}$$

Due to $\Delta\mathbf{U}(t) = \mathbf{0}$ on L_b, (33) and (34) show that $\mathbf{K}(z)$ is a holomorphic function in the z-plane except on the cracks. Moreover, (21.3) gives

$$[\Delta\mathbf{U}(t)]_{(2)} = 0 \quad \text{on } L_b + L_c, \tag{36}$$

where $[\mathbf{X}]_{(2)}$ represents taking the second row of $\mathbf{X}$.

From (33) and (36), one has

$$K_2^+(t) - K_2^-(t) = 0 \quad \text{on } L_b + L_c. \tag{37}$$

The solution of (37) is

$$K_2(z) = 0. \tag{38}$$

On the other hand, (32) can be reduced to

$$\mathbf{T}(t) = \mathbf{H}^{-1}\mathbf{K}^+(t) + \mathbf{H}^{-1}\mathbf{K}^-(t) + 2\mathbf{H}^{-1}\mathbf{B}_2^{-1}\left(\mathbf{p}^{\infty} + \overline{\mathbf{p}^{\infty}}\frac{R^2}{z^2}\right), \tag{39}$$

namely

$$\mathbf{T}(t) = \mathbf{H}^{-1}\mathbf{K}^+(t) + \mathbf{H}^{-1}\mathbf{K}^-(t) + \mathbf{g}^{\infty} + \overline{\mathbf{g}^{\infty}}\frac{R^2}{z^2}, \tag{40}$$

where

$$\mathbf{g}^{\infty} = 2\mathbf{M}\mathbf{p}^{\infty}, \quad \mathbf{M} = \mathbf{H}^{-1}\mathbf{B}_2^{-1}. \tag{41}$$

Expanding (40), and using (38), leads to

$$T_{\sigma}(t) = H_{11}^{-1}K_1^+(t) + H_{11}^{-1}K_1^-(t) + g_1^{\infty} + \overline{g_1^{\infty}}\frac{R^2}{t^2}, \tag{42}$$

$$T_{\mathrm{D}}(t) = H_{21}^{-1}K_1^+(t) + H_{21}^{-1}K_1^-(t) + g_2^{\infty} + \overline{g_2^{\infty}}\frac{R^2}{t^2}. \tag{43}$$

On the crack we have $\sigma_r^0 = 0$, i.e., $T_\sigma = 0$. Thus we obtain from (42) that

$$K_1^+(t) + K_1^-(t) = -\frac{1}{H_{11}^{-1}}\left(g_1^\infty + \overline{g_1^\infty}\frac{R^2}{t^2}\right) \quad \text{on } L_c. \tag{44}$$

Substituting (44) into (43) we have

$$T_D^0 = \frac{iR}{z}D_r^0 = (g_2^\infty - c_D g_1^\infty) + (\overline{g_2^\infty} - c_D\overline{g_1^\infty})\frac{R^2}{t^2} \quad \text{on } L_c, \tag{45}$$

where $c_D = H_{21}^{-1}/H_{11}^{-1}$, which, by using (35) and (11), can be reduced to

$$c_D = \frac{e_{15}^{(1)}\left[e_{15}^{2(2)} + c_{44}^{(2)}\varepsilon_{11}^{(2)}\right] + e_{15}^{(2)}\left[e_{15}^{2(1)} + c_{44}^{(1)}\varepsilon_{11}^{(1)}\right]}{c_{44}^{(1)}\left[e_{15}^{2(2)} + c_{44}^{(2)}\varepsilon_{11}^{(2)}\right] + c_{44}^{(2)}\left[e_{15}^{2(1)} + c_{44}^{(1)}\varepsilon_{11}^{(1)}\right]}.$$

From (45), one has

$$D_r^0(\theta) = 2\,\mathrm{Im}[e^{i\theta}(g_2^\infty - c_D g_1^\infty)] \quad \text{on } L_c. \tag{46}$$

Finally, we obtain from (46), by using (41) and (24), that

$$D_r^0(\theta) = (M_{21} - c_D M_{11})(\sigma_{32}^\infty \sin\theta + \sigma_{31}^\infty\cos\theta) + (M_{22} - c_D M_{12})(D_2^\infty\sin\theta + D_1^\infty\cos\theta). \tag{47}$$

4. Complex potentials

Inserting (40) and (45) into (27) we have

$$\mathbf{T}(t) = \mathbf{H}^{-1}\mathbf{K}^+(t) + \mathbf{H}^{-1}\mathbf{K}^-(t) + \mathbf{g}^\infty + \overline{\mathbf{g}^\infty}\frac{R^2}{z^2} = \mathbf{i}_2\left[(g_2^\infty - c_D g_1^\infty) + (\overline{g_2^\infty} - c_D\overline{g_1^\infty})\frac{R^2}{t^2}\right] \quad \text{on } L_c \tag{48}$$

which can be further reduced to

$$[\mathbf{H}^{-1}\mathbf{K}(t)]^+ + [\mathbf{H}^{-1}\mathbf{K}(t)]^- = \mathbf{P}^\infty + \overline{\mathbf{P}^\infty}\frac{R^2}{t^2}, \tag{49}$$

where

$$\mathbf{P}^\infty = \mathbf{i}_2(g_2^\infty - c_D g_1^\infty) - \mathbf{g}^\infty. \tag{50}$$

The general solution of (49) is (Muskhelishvili, 1975)

$$\mathbf{H}^{-1}\mathbf{K}(z) = \frac{1}{2}\left[\mathbf{P}^\infty + \overline{\mathbf{P}^\infty}\frac{R^2}{z^2}\right] + \frac{1}{2X(z)}\left[\mathbf{P}(z) + \frac{\boldsymbol{\beta}_{-1}}{z} + \frac{\boldsymbol{\beta}_{-2}}{z^2}\right], \tag{51}$$

where

$$\begin{aligned} &\mathbf{P}(z) = \boldsymbol{\beta}_N z^N + \boldsymbol{\beta}_{N-1}z^{N-1} + \cdots + \boldsymbol{\beta}_0, \\ &X(z) = \prod_{n=1}^{N}(z-a_n)^{1/2}(z-b_n)^{1/2}. \end{aligned} \tag{52}$$

Below we discuss the related equations to the coefficients $\boldsymbol{\beta}_n$ involved in (52.1). Near $z = 0$ one has

$$\frac{1}{X(z)} = \frac{1}{X(0)}\left[1 + \frac{z}{2}\sum_{n=1}^{N}\left(\frac{1}{a_n} + \frac{1}{b_n}\right) + \cdots\right], \tag{53}$$

where

$$X(0) = (-1)^N \prod_{n=1}^{N} \sqrt{a_n b_n}.$$

Since $\mathbf{H}^{-1}\mathbf{K}(z)$ is analytic near $z = 0$, substituting (53) into (51), and then letting the coefficients of z^{-1} and z^{-2} on the right side of (51) to be zero, we have

$$z^{-1}: \quad \boldsymbol{\beta}_{-1} + \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{a_n} + \frac{1}{b_n}\right)\boldsymbol{\beta}_{-2} = \mathbf{0}, \tag{54}$$

$$z^{-2}: \quad \overline{\mathbf{P}^{\infty}}R^2 + \frac{1}{X(0)}\boldsymbol{\beta}_{-2} = \mathbf{0}. \tag{55}$$

One obtains from (54) and (55) that

$$\begin{aligned} \boldsymbol{\beta}_{-2} &= -X(0)R^2\overline{\mathbf{P}^{\infty}}, \\ \boldsymbol{\beta}_{-1} &= -\frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{a_n} + \frac{1}{b_n}\right)\boldsymbol{\beta}_{-2}. \end{aligned} \tag{56}$$

On the other hand, when $z \to \infty$:

$$\frac{1}{X(z)} = z^{-N}\left[1 + \frac{1}{2z}\sum_{n=1}^{N}(a_n + b_n) + \mathrm{O}(z^{-2})\right]. \tag{57}$$

Substituting (57) into (51), and then noting that the coefficients of z^0 and z^{-1} on the right side of (51) are zero when $z \to \infty$, we have

$$z^0: \quad -\mathbf{P}^{\infty} + \boldsymbol{\beta}_N = \mathbf{0},$$

$$z^{-1}: \quad \boldsymbol{\beta}_{N-1} + \frac{1}{2}\sum_{n=1}^{N}(a_n + b_n)\boldsymbol{\beta}_N = \mathbf{0},$$

which results in

$$\boldsymbol{\beta}_N = \mathbf{P}^{\infty}, \tag{58}$$

$$\boldsymbol{\beta}_{N-1} = -\frac{1}{2}\sum_{n=1}^{N}(a_n + b_n)\boldsymbol{\beta}_N. \tag{59}$$

The remainder unknown coefficients can be determined from the single-valued conditions of displacement and electric potentials, such that

$$\int_{L_c} \Delta\mathbf{U}(t)\,\mathrm{d}t = \mathbf{0}. \tag{60}$$

Inserting (33) into (60) leads to

$$\int_{L_c} [\mathbf{K}^+(t) - \mathbf{K}^-(t)]\,\mathrm{d}t = \mathbf{0},$$

namely

$$\int_{L_c} [\mathbf{H}^{-1}\mathbf{K}^+(t) - \mathbf{H}^{-1}\mathbf{K}^-(t)]\,\mathrm{d}t = \mathbf{0}. \tag{61}$$

Substituting (51) into (61) produces

$$\int_{L_c} \frac{1}{X^+(t)} \left[\mathbf{P}(t) + \frac{\boldsymbol{\beta}_{-1}}{t} + \frac{\boldsymbol{\beta}_{-2}}{t^2} \right] \mathrm{d}t = \mathbf{0}. \tag{62}$$

After (51) is completely determined, $\mathbf{F}_{k0}(z)$ can be found by using (34). With $\mathbf{F}_{k0}(z)$ all field variables in the matrix and inhomogeneity can be readily determined.

5. Field intensity factors

For any crack l_n, define the field intensity factors at the crack tip $z = b_n = R\mathrm{e}^{\mathrm{i}\theta_\mathrm{b}}$ as

$$\mathbf{k}(b_n) = [k_\sigma, k_\mathrm{D}]^\mathrm{T} = \lim_{\rho \to 0} \sqrt{2\pi\rho}[\sigma_{3r}, D_r]^\mathrm{T}, \quad k_\mathrm{E}(b_n) = \lim_{\rho \to 0} \sqrt{2\pi\rho} E_r, \tag{63}$$

where ρ means the distance from the crack tip along the interface; k_σ, k_D and k_E are the intensity factors of stress, electric displacement and electric field, respectively.

From (13) one has

$$[\sigma_{3r}, D_r]^\mathrm{T} = 2\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta}\mathbf{BF}(z)], \quad E_r = -2\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta}\Omega_2(z)]. \tag{64}$$

Noting $\rho = R\,\mathrm{d}\theta$, and

$$\mathbf{BF}^\mathrm{s}(z) = \mathbf{BF}_0^\mathrm{s}(z) = \mathbf{H}^{-1}\mathbf{K}(z), \quad \Omega_2^\mathrm{s}(z) = [\mathbf{F}^\mathrm{s}(z)]_{(2)} = [\mathbf{B}^{-1}\mathbf{H}^{-1}\mathbf{K}(z)]_{(2)},$$

where the superscript s stands for the singular principle parts of functions, and then inserting (64) into (63) produces

$$\mathbf{k}(b_n) = 2\sqrt{2\pi R}\,\mathrm{Im}\left[\lim_{\theta \to \theta_\mathrm{b}} (\theta - \theta_\mathrm{b})^{1/2} \mathrm{e}^{\mathrm{i}\theta}\mathbf{H}^{-1}\mathbf{K}(z) \right],$$

$$k_\mathrm{E}^{(j)}(b_n) = -2\sqrt{2\pi R}\,\mathrm{Im}\left[\boldsymbol{\Lambda}^{(j)} \lim_{\theta \to \theta_\mathrm{b}} (\theta - \theta_\mathrm{b})^{1/2} \mathrm{e}^{\mathrm{i}\theta}\mathbf{H}^{-1}\mathbf{K}(z) \right]_{(2)}, \tag{65}$$

where $k_\mathrm{E}^{(j)}$ is the intensity factor of electric field when z approaches into the crack tip from within the inhomogeneity ($j = 1$) or the matrix ($j = 2$), and $\boldsymbol{\Lambda}^{(j)} = \mathbf{B}_j^{-1}$.

Using (38), (65) can be rewritten as

$$k_\sigma(b_n) = H_{11}^{-1}\phi(\theta_\mathrm{b}), \quad k_\mathrm{D}(b_n) = H_{21}^{-1}\phi(\theta_\mathrm{b}), \quad k_\mathrm{E}^{(j)}(b_n) = -\left[\Lambda_{21}^{(j)} H_{11}^{-1} + \Lambda_{22}^{(j)} H_{21}^{-1} \right] \phi(\theta_\mathrm{b}), \tag{66}$$

where

$$\phi(\theta_\mathrm{b}) = 2\sqrt{2\pi R}\,\mathrm{Im}\left[\lim_{\theta \to \theta_\mathrm{b}} (\theta - \theta_\mathrm{b})^{1/2} \mathrm{e}^{\mathrm{i}\theta} K_1(z) \right]. \tag{67}$$

From (66) one has

$$k_\mathrm{D} = c_\mathrm{D} k_\sigma, \quad k_\mathrm{E}^{(j)} = c_\mathrm{E}^{(j)} k_\sigma, \tag{68}$$

where

$$c_\mathrm{E}^{(j)} = -[\Lambda_{21}^{(j)} + c_\mathrm{D}\Lambda_{22}^{(j)}]. \tag{69}$$

It can be shown from (69) that

$$c_{\mathrm{E}}^{(1)} = -c_{\mathrm{E}}^{(2)} = \frac{e_{15}^{(1)} c_{44}^{(2)} - e_{15}^{(2)} c_{44}^{(1)}}{c_{44}^{(1)}\left[e_{15}^{2(2)} + c_{44}^{(2)} \varepsilon_{11}^{(2)}\right] + c_{44}^{(2)}\left[e_{15}^{2(1)} + c_{44}^{(1)} \varepsilon_{11}^{(1)}\right]}. \tag{70}$$

Eqs. (68) and (70) show that k_{D} and k_{E} are always dependent on k_σ, and moreover the electric field is jumpy ahead of the crack tip.

Furthermore, substituting (51) into (65.1) leads to the general expression of the field factor vector as

$$\mathbf{k}(b_n) = \sqrt{2\pi R}\,\mathrm{Im}\left\langle \lim_{\theta \to \theta_{\mathrm{b}}} (\theta - \theta_{\mathrm{b}})^{1/2} \frac{\mathrm{e}^{\mathrm{i}\theta}}{X(z)} \left[\mathbf{P}(z) + \frac{\boldsymbol{\beta}_{-1}}{z} + \frac{\boldsymbol{\beta}_{-2}}{z^2}\right] \right\rangle, \quad z = R\mathrm{e}^{\mathrm{i}\theta}. \tag{71}$$

Similarly, at $x = a_n$ one has

$$\mathbf{k}(a_n) = \sqrt{2\pi R}\,\mathrm{Im}\left\langle \lim_{\theta \to \theta_a} (\theta_a - \theta)^{1/2} \frac{\mathrm{e}^{\mathrm{i}\theta}}{X(z)} \left[\mathbf{P}(z) + \frac{\boldsymbol{\beta}_{-1}}{z} + \frac{\boldsymbol{\beta}_{-2}}{z^2}\right] \right\rangle. \tag{72}$$

6. Examples

6.1. The solution to a permeable crack

For the case shown in Fig. 2, we have

$$\begin{aligned} &a_1 = R\mathrm{e}^{-\mathrm{i}\theta_0}, \quad b_1 = R\mathrm{e}^{\mathrm{i}\theta_0}, \quad X(z) = \sqrt{z^2 - 2zR\cos\theta_0 + R^2}, \\ &X(0) = -\sqrt{a_1 b_1} = -R, \quad \mathbf{P}(z) = \boldsymbol{\beta}_1 z + \boldsymbol{\beta}_0. \end{aligned} \tag{73}$$

From (73), (56), (58), (59) one can obtain

$$\begin{aligned} &\boldsymbol{\beta}_1 = \mathbf{P}^\infty, \quad \boldsymbol{\beta}_0 = -R\cos\theta_0 \mathbf{P}^\infty, \\ &\boldsymbol{\beta}_{-2} = R^3 \overline{\mathbf{P}^\infty}, \quad \boldsymbol{\beta}_{-1} = -R^2 \cos\theta_0 \overline{\mathbf{P}^\infty}. \end{aligned} \tag{74}$$

Substituting (73) with (74) into (71), and then using the following identities:

$$X(\theta) = R\sqrt{\mathrm{e}^{2\mathrm{i}\vartheta} - 2\mathrm{e}^{\mathrm{i}\theta}\cos\theta_0 + 1} = \mathrm{e}^{\mathrm{i}\theta/2} R\sqrt{\mathrm{e}^{\mathrm{i}\vartheta} - 2\cos\theta_0 + \mathrm{e}^{\mathrm{i}\theta}} = \mathrm{e}^{\mathrm{i}\theta/2} R\sqrt{2}\sqrt{\cos\theta - \cos\theta_0},$$

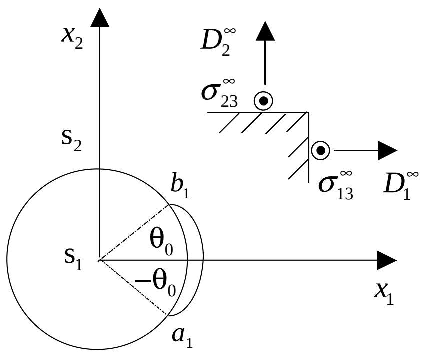

Fig. 2. The case of an interface crack.

and

$$\cos\theta = \cos[\theta_0 + (\theta - \theta_0)] = \cos\theta_0 - (\theta - \theta_0)\sin\theta_0 + \cdots,$$

we obtain

$$\mathbf{k}(b_1) = 2\sqrt{\pi R\sin\theta_0}\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta_0/2}\mathbf{P}^\infty]. \tag{75}$$

Similarly one has

$$\mathbf{k}(a_1) = -2\sqrt{\pi R\sin\theta_0}\,\mathrm{Im}[\mathrm{e}^{-\mathrm{i}\theta_0/2}\mathbf{P}^\infty], \tag{76}$$

in which $\mathbf{P}^\infty$, defined by (50), can be reduced to

$$\mathbf{P}^\infty = -g_1^\infty\mathbf{k}_0, \tag{77}$$

where

$$\mathbf{k}_0 = [1, c_\mathrm{D}]^\mathrm{T}, \quad g_1^\infty = M_{11}(\sigma_{32}^\infty + \mathrm{i}\sigma_{31}^\infty) + M_{12}(D_2^\infty + \mathrm{i}D_1^\infty). \tag{78}$$

Substituting (77) into (75) and (76) results in

$$\mathbf{k}(b_1) = -2\sqrt{\pi R\sin\theta_0}\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta_0/2}g_1^\infty]\mathbf{k}_0, \tag{79}$$

$$\mathbf{k}(a_1) = +2\sqrt{\pi R\sin\theta_0}\,\mathrm{Im}[\mathrm{e}^{-\mathrm{i}\theta_0/2}g_1^\infty]\mathbf{k}_0. \tag{80}$$

Inserting (78.1) into (79) and (80) leads to

$$\mathbf{k}(b_1) = -2\sqrt{\pi R\sin\theta_0}\left[M_{11}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} + \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right) + M_{12}\left(D_2^\infty\sin\frac{\theta_0}{2} + D_1^\infty\cos\frac{\theta_0}{2}\right)\right]\mathbf{k}_0, \tag{81}$$

$$\mathbf{k}(a_1) = +2\sqrt{\pi R\sin\theta_0}\left[M_{11}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} - \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right) + M_{12}\left(D_2^\infty\sin\frac{\theta_0}{2} - D_1^\infty\cos\frac{\theta_0}{2}\right)\right]\mathbf{k}_0. \tag{82}$$

For the case of purely elastic materials,

$$B_1 = c_{44}^{(1)}, \quad B_2 = c_{44}^{(2)}, \quad M_{11} = \Gamma/(1+\Gamma), \quad M_{12} = 0,$$

where $\Gamma = c_{44}^{(1)}/c_{44}^{(2)}$, and (81) and (82) degenerates into

$$\begin{aligned} k_\sigma(b_1) &= -\frac{2\Gamma}{1+\Gamma}\sqrt{\pi R\sin\theta_0}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} + \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right), \\ k_\sigma(a_1) &= \frac{2\Gamma}{1+\Gamma}\sqrt{\pi R\sin\theta_0}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} - \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right), \end{aligned} \tag{83}$$

which is consistent with that of Gong and Meguid (1994).

For the case of a homogeneous piezoelectric material, $\mathbf{M} = \mathbf{I}/2$, and we have from (81) and (82) that

$$\begin{aligned} \mathbf{k}(b_1) &= -\sqrt{\pi R\sin\theta_0}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} + \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right)\mathbf{k}_0, \\ \mathbf{k}(a_1) &= \sqrt{\pi R\sin\theta_0}\left(\sigma_{32}^\infty\sin\frac{\theta_0}{2} - \sigma_{31}^\infty\cos\frac{\theta_0}{2}\right)\mathbf{k}_0, \end{aligned} \tag{84}$$

which implies that in this case, the intensity factors are not dependent on the applied electric loadings.

6.2. The solution to two permeable cracks

For the case shown in Fig. 3, one has

$$a_1 = R\mathrm{e}^{-\mathrm{i}\theta_0}, \quad b_1 = R\mathrm{e}^{\mathrm{i}\theta_0}, \quad a_2 = -a_1, \quad b_2 = -b_1, \quad \mathbf{P}(z) = \boldsymbol{\beta}_2 z^2 + \boldsymbol{\beta}_1 z + \boldsymbol{\beta}_0,$$
$$X(z) = \sqrt{z^4 - 2z^2R^2\cos 2\theta_0 + R^4}, \quad X(0) = R^2 \tag{85}$$

and

$$\boldsymbol{\beta}_2 = \mathbf{P}^{\infty}, \quad \boldsymbol{\beta}_1 = \mathbf{0},$$
$$\boldsymbol{\beta}_{-2} = -R^4\overline{\mathbf{P}^{\infty}}, \quad \boldsymbol{\beta}_{-1} = \mathbf{0}. \tag{86}$$

Substituting (85) with (86) into (71), and then using:

$$X(\theta) = \sqrt{2}R^2\mathrm{e}^{\mathrm{i}\theta}\sqrt{\cos 2\theta_0 - \cos 2\theta}, \tag{87}$$

we obtain

$$\mathbf{k}(b_1) = \sqrt{2\pi R}\,\mathrm{Im}\left\langle \frac{1}{2R^2\mathrm{i}\sqrt{\sin 2\theta_0}}\left[\boldsymbol{\beta}_2 z_\mathrm{b}^2 + \boldsymbol{\beta}_0 + \frac{\boldsymbol{\beta}_{-2}}{z_\mathrm{b}^2}\right]\right\rangle. \tag{88}$$

Eq. (86) shows that

$$\boldsymbol{\beta}_2 z_\mathrm{b}^2 + \frac{\boldsymbol{\beta}_{-2}}{z_\mathrm{b}^2},$$

is purely imaginary, and thus (88) becomes

$$\mathbf{k}(b_1) = -\frac{\sqrt{\pi R}}{\sqrt{2\sin 2\theta_0}R^2}\,\mathrm{Re}[\boldsymbol{\beta}_0]. \tag{89}$$

It can be shown (see Appendix A)

$$\boldsymbol{\beta}_0 = \cos 2\theta_0 R^2\overline{\mathbf{P}^{\infty}} - R^2\mathbf{P}^{\infty}. \tag{90}$$

Substituting (90) into (89) leads to

$$\mathbf{k}(b_1) = -\frac{\sqrt{\pi R}}{\sqrt{2\sin 2\theta_0}}(\cos 2\theta_0 - 1)\,\mathrm{Re}[\mathbf{P}^{\infty}]. \tag{91}$$

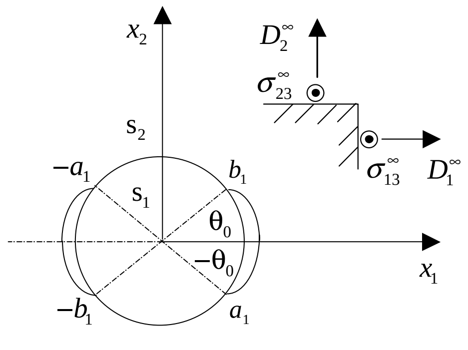

Fig. 3. The case of two interface cracks.

Using (77) and (78.2), (91) can be further reduced to

$$\mathbf{k}(b_1) = -\sqrt{2\pi R}\frac{\sin^2\theta_0}{\sqrt{\sin 2\theta_0}}(M_{11}\sigma_{32}^{\infty} + M_{12}D_2^{\infty})\mathbf{k}_0. \tag{92}$$

6.3. The solution for the case of impermeable cracks

If the cracks are impermeable, one has $D_r^0 = 0$, which is equivalent to letting the coefficient of $\mathbf{i}_2$ be zero. Considering this we have from (50) that

$$\mathbf{P}^{\infty} = -\mathbf{g}^{\infty}. \tag{93}$$

Thus, the corresponding intensity factors, from (75) and (91), are

$$\mathbf{k}(b_1) = -\sqrt{\pi R\sin\theta_0}\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta_0/2}\mathbf{g}^{\infty}], \tag{94}$$

for the case of an impermeable crack, and

$$\mathbf{k}(b_1) = \frac{\sqrt{\pi R}}{\sqrt{2\sin 2\theta_0}}(\cos 2\theta_0 - 1)\,\mathrm{Re}[\mathbf{g}^{\infty}], \tag{95}$$

for the case of two impermeable cracks.

Substituting (41) with (24) into (94) and (95) we obtain

$$\mathbf{k}(b_1) = -2\sqrt{\pi R\sin\theta_0}\left[\mathbf{M}\left(\sigma_{32}^{\infty}\sin\frac{\theta_0}{2} + \sigma_{31}^{\infty}\cos\frac{\theta_0}{2}, D_2^{\infty}\sin\frac{\theta_0}{2} + D_1^{\infty}\cos\frac{\theta_0}{2}\right)^{\mathrm{T}}\right],$$

for the case of a crack, and

$$\mathbf{k}(b_1) = -\sqrt{2\pi R}\frac{\sin^2\theta_0}{\sqrt{\sin 2\theta_0}}\mathbf{M}(\sigma_{32}^{\infty}, D_2^{\infty})^{\mathrm{T}},$$

for the case of two cracks.

6.4. The solution for the case of a completely bonded inhomogeneity

In this case, the generalized displacement is continuous on the whole circular rim, and thus one has from (33) that

$$\mathbf{K}^{+}(t) - \mathbf{K}^{-}(t) = 0 \quad \text{on } L. \tag{96}$$

The solution of (96) is

$$\mathbf{K}(z) = \mathbf{K}(\infty) = \mathbf{0}. \tag{97}$$

Substituting (97) into (34) leads to

$$\begin{aligned}&\mathbf{B}_1\mathbf{F}_{10}(z) = 2\mathbf{M}\mathbf{p}^{\infty}, \quad z\in s_1,\\ &\mathbf{B}_2\mathbf{F}_{20}(z) = \mathbf{H}^{-1}(\mathbf{B}_1^{-1} - \mathbf{B}_2^{-1})\overline{\mathbf{p}^{\infty}}\frac{R^2}{z^2}, \quad z\in s_2.\end{aligned} \tag{98}$$

Eq. (98) shows that all the field variables are uniform inside the inhomogeneity, and they can be expressed, from (9), as

$$\begin{Bmatrix}\sigma_{32}^0 + \mathrm{i}\sigma_{31}^0\\ D_2^0 + \mathrm{i}D_1^0\end{Bmatrix} = 2\mathbf{B}_1\mathbf{F}_{10}(z), \quad E_2^0 + \mathrm{i}E_2^0 = -2[\mathbf{F}_{10}(z)]_{(2)}. \tag{99}$$

Using (98.1) and (99) we obtain the relation between the remotely applied fields and those inside the inhomogeneity as

$$\left\{ \begin{array}{c} \sigma_{32}^0 + \mathrm{i}\sigma_{31}^0 \\ D_2^0 + \mathrm{i}D_1^0 \end{array} \right\} = 2\mathbf{M} \left\{ \begin{array}{c} \sigma_{32}^\infty + \mathrm{i}\sigma_{31}^\infty \\ D_2^\infty + \mathrm{i}D_1^\infty \end{array} \right\}, \tag{100}$$

$$E_2^0 = -2[W_{21}\sigma_{32}^\infty + W_{22}D_2^\infty], \quad E_1^0 = -2[W_{21}\sigma_{31}^\infty + W_{22}D_1^\infty], \tag{101}$$

where $\mathbf{W} = (\mathbf{B}_1 + \mathbf{B}_2)^{-1}$.

For the case of an infinite isotropic matrix containing a piezoelectric fibre, letting $\mathbf{B}_2 = c_{44}^{(2)}$ and noting from (41.2) that

$$\mathbf{M} = (\mathbf{B}_2\mathbf{H})^{-1} = (\mathbf{I} + c_{44}^{(2)}\mathbf{B}_1^{-1})^{-1},$$

Eq. (100) becomes

$$\left(\mathbf{B}_1 + c_{44}^{(2)}\right) \left\{ \begin{array}{c} \sigma_{32}^0 + \mathrm{i}\sigma_{31}^0 \\ D_2^0 + \mathrm{i}D_1^0 \end{array} \right\} = 2\mathbf{B}_1 \left\{ \begin{array}{c} \sigma_{32}^\infty + \mathrm{i}\sigma_{31}^\infty \\ D_2^\infty + \mathrm{i}D_1^\infty \end{array} \right\}. \tag{102}$$

7. Conclusions

We study anti-plane strain problems of arc-shaped interfacial cracks between two piezoelectric materials by using the Muskhelishvili's theory. The presented solutions are not only valid to permeable and impermeable crack models, but also are very concise. This makes it easy to observe the mechanical-electric coupling effects. It is found that for permeable interfacial cracks, the field singularities are dependent on the applied mechanical load as well as the applied electric load, which is different from the results in the case of straight cracks. However, for permeable cracks in a homogeneous material the field intensity factors are still independent of the applied electric load.

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Appendix A. Derivation of (90)

Using (85) and (86), (62) can be expressed as

$$\int_a^b \frac{1}{X^+(t)} \left[\boldsymbol{\beta}_2 t^2 + \boldsymbol{\beta}_0 + \frac{\boldsymbol{\beta}_{-2}}{t^2} \right] \mathrm{d}t = \mathbf{0}, \quad t = R\mathrm{e}^{\mathrm{i}\theta}, \tag{A.1}$$

in which $X^+(t)$, i.e., (87), has the form

$$X^+(t) = 2R^2\mathrm{e}^{\mathrm{i}\theta}\sqrt{\sin^2\theta_0 - \sin^2\theta} = 2R^2\sin\theta_0\mathrm{e}^{\mathrm{i}\theta}\varDelta, \tag{A.2}$$

where

$$\varDelta = \sqrt{1-\lambda^2\sin^2\theta}, \quad \lambda = \frac{1}{\sin\theta_0} > 1.$$

Substituting (A.2) into (A.1) yields

$$\mathbf{P}^{\infty}R^2 I_1 + I_{-1}\boldsymbol{\beta}_0 - \overline{\mathbf{P}^{\infty}}R^2 I_{-3} = \mathbf{0}, \tag{A.3}$$

where

$$I_n = \int_{-\theta_0}^{+\theta_0} \frac{e^{in\theta}}{\varDelta}\,d\theta, \quad n = \pm 1, -3. \tag{A.4}$$

Eq. (A.4) can be further reduced to

$$I_n = 2\int_0^{\theta_0} \frac{\cos n\theta}{\varDelta}\,d\theta. \tag{A.5}$$

Noting $I_1 = I_{-1}$, one obtains from (A.3) that

$$\boldsymbol{\beta}_0 = \frac{I_{-3}}{I_{-1}}\overline{\mathbf{P}^{\infty}}R^2 - \mathbf{P}^{\infty}R^2. \tag{A.6}$$

Using the following identities:

$$\frac{I_{-3}}{I_{-1}} = \frac{\int_0^{\theta_0}\frac{\cos 3\theta}{\varDelta}\,d\theta}{\int_0^{\theta_0}\frac{\cos\theta}{\varDelta}\,d\theta} = \frac{\int_0^{\theta_0}\frac{4\cos^3\theta - 3\cos\theta}{\varDelta}\,d\theta}{\int_0^{\theta_0}\frac{\cos\theta}{\varDelta}\,d\theta} = \frac{4\int_0^{\theta_0}\frac{\cos^3\theta}{\varDelta}\,d\theta}{\int_0^{\theta_0}\frac{\cos\theta}{\varDelta}\,d\theta} - 3, \tag{A.7}$$

and integration results (Prudnikov et al., 1990):

$$\int_0^{\theta_0}\frac{\cos\theta}{\varDelta}\,d\theta = \frac{1}{\lambda}\arcsin(\lambda\sin\theta_0),$$

$$\int_0^{\theta_0}\frac{\cos^3\theta}{\varDelta}\,d\theta = \varDelta_0\frac{\sin\theta_0}{2\lambda^2} + \frac{2\lambda^2-1}{2\lambda^3}\arcsin(\lambda\sin\theta_0),$$

where

$$\varDelta_0 = \sqrt{1-\lambda^2\sin^2\theta_0} = 0,$$

we obtain from (A.7) that

$$\frac{I_{-3}}{I_{-1}} = 4\left(1 - \frac{1}{2\lambda^2}\right) - 3 = \cos 2\theta_0. \tag{A.8}$$

Substituting (A.8) into (A.6) finally gives

$$\boldsymbol{\beta}_0 = \cos 2\theta_0 R^2\overline{\mathbf{P}^{\infty}} - R^2\mathbf{P}^{\infty}.$$

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